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A SIMPLIFIED PROOF OF THE DISTRIBUTIVE LAW OF MULTIPLICATION GIVEN IN HILBERT'S "THE FOUNDATIONS OF GEOMETRY."

By WILLIAM E. ROTH, University of Illinois.

Professor Hilbert has developed an algebra of segments in his *Foundations of Geometry*¹ on the following assumptions:

I. The axioms of connection:

1. Two distinct points A and B always completely determine a straight line.
2. Any two distinct points of a straight line completely determine that line.

II. The axioms of order.

III. The parallel axiom.

IV. A special form of Desargue's theorem.

In the proof which follows, the axioms of order are not used and, therefore, need not be stated; the parallel axiom is that of Euclid; and, of the special form of Desargue's theorem used by Hilbert, only the part given below is applied in the demonstration:

If two triangles are so situated in a plane that the straight lines joining the homologous vertices intersect in a common point, or are parallel to one another, and, furthermore, if two pairs of homologous sides are parallel to each other, then the third sides of the two triangles are parallel to each other.

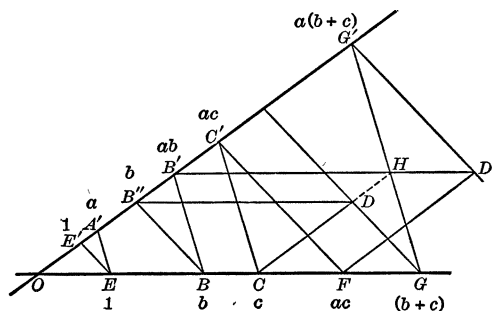
On the basis of the above assumptions, Hilbert defines both addition and multiplication of segments by constructions which will be given below. However, the associative law of multiplication is not established, and so it is desirable to establish the distributive law in two forms, namely

$$a(b + c) = ab + ac,$$

and

$$(a + b)c = ac + bc.$$

The present article aims to present for the first of these a simpler proof than that given by Dr. von Schaper in Hilbert's book.



¹ Translation by E. J. Townsend, pages 79-99.

Constructions: Take on one of two straight lines meeting in O

$$1 = OE, \quad b = OB, \quad c = OC,$$

and on the other

$$1 = OE', \quad a = OA'.$$

Draw EA' and parallel to it draw BB' and CC' , which meet OE' in the points B' and C' respectively. Then by Hilbert's definition of multiplication (*l. c.*, page 81),

$$OB' = ab \text{ and } OC' = ac.$$

Draw the unit line EE' and parallel to it draw BB'' and $C'F$, which respectively meet OE' in B'' and OE in F . Then

$$OB'' = b \text{ and } OF = ac.$$

Next construct $B''D$ and $B'D'$ parallel to OE , and CD and FD' parallel to OE' ; the points D and D' are uniquely determined. A straight line through D parallel to EE' cuts OE in a point G , such that by the definition of addition of segments (Hilbert, *l. c.*, page 81),

$$OG = b + c.$$

Now take GG' parallel to EA' ; so again by the definition of multiplication

$$OG' = a(b + c). \quad (1)$$

If we were to draw through D' a straight line parallel to EE' , it would cut OE' in a point defining the segment $ab + ac$, but instead of making this construction, we shall simply draw the straight line $D'G'$ and prove that it has the above property.

Proof: Denote the point of intersection of GG' and $B'D'$ by H . Connect D and H by a straight line. In the triangles $BB''B'$ and GDH , we have the homologous sides BB'' and BB' parallel to GD and GH respectively, and their corresponding vertices are connected by the parallel lines BG , $B''D$, and $B'H$. Then by the part of Desargue's theorem given above, DH is parallel to $B'B''$. But by construction CD is parallel to OE' ; so the points C , D and H are collinear because of the parallel axiom. Therefore, in the triangles $CC'F$ and $HG'D'$, the following relations hold: CC' and CF are parallel to HG' and HD' respectively; also the lines joining the corresponding vertices are mutually parallel; consequently $G'D'$ is parallel to $C'F$ by Desargue's theorem. Then $G'D'$ is parallel to the unit line EE' . Then from the definition of addition, we have

$$OG' = ab + ac. \quad (2)$$

Then through equation (1) we get immediately

$$a(b + c) = ab + ac.$$

Thus the first form of the distributive law of multiplication is demonstrated on the basis of the given assumptions.

The above proof is simpler than that of Dr. von Schaper in that it requires only two applications of the second part of Desargue's theorem; whereas, the latter requires three applications of the first part of this theorem and five of the second part to complete it.

THE DERIVATIVE OF THE LOGARITHM.

By M. B. PORTER, University of Texas.

That the problem of deriving the logarithm presents pedagogic difficulties is sufficiently evident to any one who turns the pages of the texts on the calculus. A great many of these content themselves with showing that $[1 + (1/n)]^n$, n positive and integral, approaches $\Sigma(1/n!)$ as a limit as n becomes infinite and hence that, if $\log x \doteq \log e$ as $x \doteq e +$, the right hand incremental ratio approaches the limit $(1/x) \cdot \log_e a$ when Δx approaches zero over a certain denumerable point set.¹ Some show that this limit is the same over any point set to the right or left of x , though all assume the continuity of $\log x$. The mechanism of this proof involves the binomial theorem for positive integral exponents, simple convergence tests, and obvious inequalities. The main criticisms that can be urged against such proofs is that they are incomplete, that the binomial theorem has usually been proved by an incomplete induction, and that the proof involves many different steps. The steps are simple in themselves, but after all almost as much is assumed as is proved.

In the first edition of Vallée-Poussin's *Cours* an interesting proof of these results is obtained by means of the elementary inequality $a^{n+1} > 1 + (n+1)(a-1)$, followed by the substitution of $[1 + \omega/(n+1)] \div [1 + (\omega/n)]$ for a .² Here, while the steps are all elementary, the obvious artificiality of the whole process unfits it for elementary instruction; the substitution is one that the student would never invent for himself or remember. On the other hand, the Davis-Hedrick *Calculus*, frankly recognizing the unconvincing character of elaborate proof as well as its incompleteness, for the immature mind of the average beginner, makes a stronger appeal to intuition and thus obtains a greater vividness of effect by a sharper, quicker attack and produces quite as satisfactory a state of mental *bien être* on the part of the youthful and uncritical student as that obtained by the more tiresome process, thus following the safe pedagogic principle that it is not worth while to bother the student with details of proof which he cannot understand or at least whose necessity he does not appreciate.

The question of the continuity of the logarithm can be treated by constructing the values of the logarithm function by the insertion of a series of arithmetic and geometric means—the method used by Briggs in the calculation of his tables.

To many teachers of the calculus it seems desirable to put in the hands of the student a simple outline of a proof, which he can fill in, whereby the existence of

¹ Osborne's *Calculus*, revised ed., pp. 9–10.

² See Granville's *Calculus*, p. 31, where the proof is reproduced.